DSP

29/10/2015 الخميس — د. عرفة — محاضرة $\boxed{5}$

Discrete Fourier Transform (D.F.T)

If we want to analyze any signal, we must evaluate some parameters as:-

1- freq. Content. ⟶ Both are analyze in freq. domain
2- Power and Energy Density.
3- Amplitude.
4- Periodicity.

we want to convert signals from time domain to freq. domain, to study some parameters of the signals, as (freq. Content and Power and Energy density), which can not be studied in time domain.

* Some applications depend on the analysis of the Signals on freq. domain as filter design.

from Laplace Transform

$$\underset{\text{time domain}}{X(t)} \xrightarrow{L.T} \underset{S\text{-domain}}{X(S)}$$

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

- Cont. fourier tranrform (FT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$\Downarrow$ discretization

$X(K)$ Discrete fourier transform (DFT)

This deosn't work for digital Systems

we need Discretization

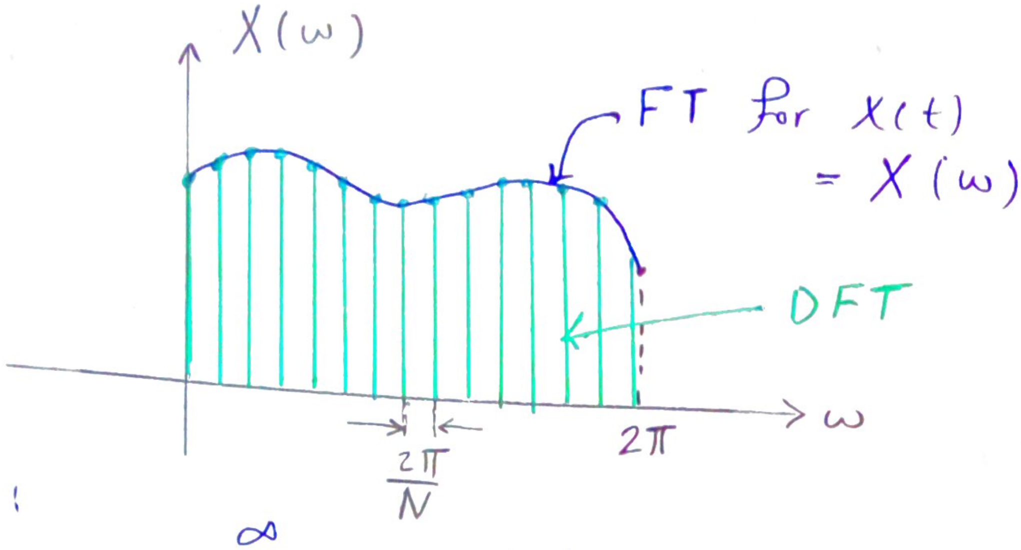
$x(t) \xrightarrow{FT} X(\omega)$

$x(n) \xrightarrow{DFT} X(K)$

* if $x(t)$ is periodic Signal, then the FT of $x(t)$ $(X(\omega))$ is also Periodic. And it's periodic every 2π interval of freq. $\Leftarrow$ F.T مضمون

* if we assume the signals we deal with are Periodic.

$\Longrightarrow$ Turn over



FT:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

assume $X(\omega)$ is dicretized to N samples

$$\omega \longrightarrow \frac{2\pi}{N} K$$

K: the sample number for DFT.

$K \Rightarrow [0 \rightarrow N-1]$

N: No. of samples for DFT.

DFT:-

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn}$$

n: The Sample no. for discrete time sequence

$x(n)$, $n = 0, 1, 2, \cdots N-1$

$\Longrightarrow$ Turn Over

Ex: $x(n) = \{0, 1, 2, 3\}$, find the 4-point DFT of $x(n)$, given $N=4$

$n = 0, 1, 2, \ldots, N-1$
$K = 0, 1, 2, \ldots, N-1$

$$K = 0 \Rightarrow X(0) = \sum_{n=0}^{3} x(n)$$

$$= x(0) + x(1) + x(2) + x(3) = 6$$

$$K = 1 \Rightarrow X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{4}(1)\times n}$$

$$= x(0) + x(1)\,e^{-j\pi/2} + x(2)\,e^{-j\pi} + x(3)\,e^{-j3\pi/2}$$

$$= 0 + e^{-j\pi/2} + 2e^{-j\pi} + 3e^{-j3\pi/2}$$

$$= \underbrace{\cos(90)}_{0} - j\underbrace{\sin(90)}_{1} + 2(\underbrace{\cos(2\pi)}_{1} - j\underbrace{\sin(2\pi)}_{0})$$
$$+ 3(\underbrace{\cos(3\pi/2)}_{0} - j\underbrace{\sin(\tfrac{3\pi}{2})}_{-1})$$

$$= -2 + j2$$

$$K = 2 \Rightarrow X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$= x(\underbrace{0}_{0}) + x(\underbrace{1}_{1})\,e^{-j\pi} + x(\underbrace{2}_{2})\,e^{-j2\pi} + x(\underbrace{3}_{3})\,e^{-j3\pi}$$

$$= e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi}$$

$$= \underbrace{\cos\pi}_{-1} - j\underbrace{\sin\pi}_{0} + 2(\underbrace{\cos(2\pi)}_{1} - j\underbrace{\sin(2\pi)}_{0})$$
$$+ 3(\underbrace{\cos(3\pi)}_{-1} - j\underbrace{\sin(3\pi)}_{0})$$

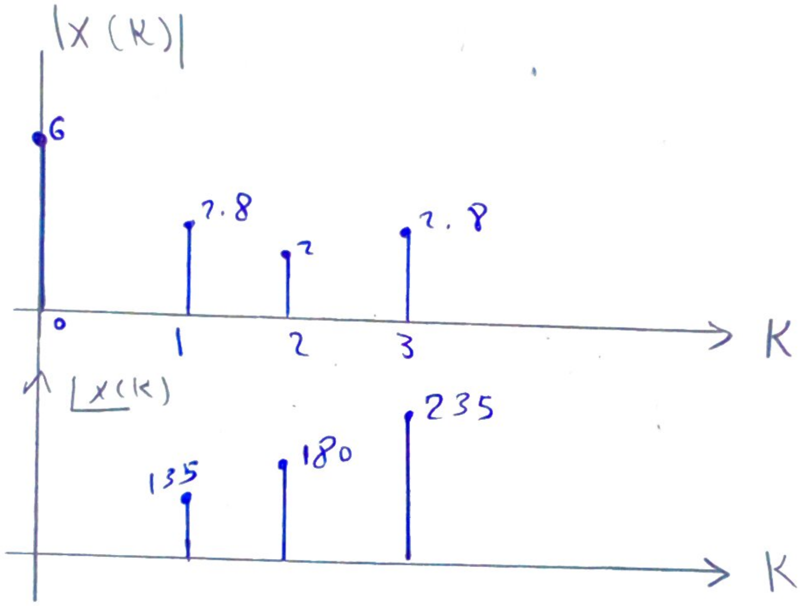
$$= -1 + 2 - 3 = -2$$

$$K = 3 \Rightarrow X(3) = -2 - 2j$$

$$X(K) = \{6,\ -2+j2,\ -2,\ -2-j2\}$$

$$|X(K)| = \{6,\ 2\sqrt{2},\ 2,\ 2\sqrt{2}\}$$

$$\angle X(K) = \{0,\ 135^\circ,\ 180^\circ,\ 225^\circ)$$



Max K for exam 3 or 4

هناك طريقة أسرع من خلال Matrix

$$x(n) \xrightarrow{DFT} X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\Rightarrow X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$K = 0, 1, 2, \cdots, N-1$$

$$n = 0, 1, 2, \cdots, N-1$$

$$X_N = \begin{pmatrix} X\ (K=0) \\ X\ (K=1) \\ \vdots \\ X\ (N-1) \end{pmatrix} , \quad x_N = \begin{pmatrix} x\ (n=0) \\ x\ (n=1) \\ \vdots \\ x\ (n=n-1) \end{pmatrix}$$

$$X_N = \begin{array}{c} \\ K=0 \\ K=1 \\ \vdots \\ K=N-1 \end{array} \begin{array}{c} n=0 \quad n=1 \quad \cdots \quad n=N-1 \\ \left[\begin{matrix} \\ W_N^{Kn} \\ \\ \end{matrix} \right]_{N \times N} \end{array} x_N$$

Ex: $N = 4$

$$\left[W_4^{Kn} \right] = \begin{array}{c} \\ K=0 \\ K=1 \\ K=2 \\ K=3 \end{array} \begin{array}{c} n=0 \qquad n=1 \qquad n=2 \qquad n=3 \\ \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & 4_4^6 & W_4^9 \end{pmatrix} \end{array}$$

$$W_4 = e^{-j\frac{2\pi}{4}} = \cos(90) - j\sin(90) = -j$$

$$W_4^0 = 1$$

$$W_4^2 = e^{-j\pi} = \cos(180) - j\sin(180) = -1$$

$$W_4^3 = e^{-j\frac{3}{2}\pi} = \cos(270) - j\sin(270) = j$$

$$W_4^4 = W_4^0 = 1$$

$$W_4^6 = W_4^2 = -1$$

$$W_4^9 = W_4^1 = -j$$

$$W_N^N = e^{\frac{-j2\pi N}{N}} = e^{-j2\pi} = 1$$

$$W_4^6 = W_4^2 \cdot \underbrace{W_4^4}_{1}$$

$$W_4^9 = \underbrace{W_4^4}_{1} \cdot \underbrace{W_4^4}_{1} \cdot W_4^1$$

$\Rightarrow$ Turn over

$$[W_4] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

DFT for $x(n)$: $(N=4)$

$$X_4 = [W_4]\, x_4$$

$X(K=0)$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix}$$

$X(K=4)$

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for $\boxed{N=3}$

$$[W_3]_{3\times 3} = \begin{array}{c} \\ K=0 \\ K=1 \\ K=2 \end{array} \begin{array}{c} \begin{array}{ccc} n=0 & n=1 & n=2 \end{array} \\ \begin{pmatrix} w_3^0 & w_3^0 & w_3^0 \\ w_3^0 & w_3^1 & w_3^2 \\ w_3^0 & w_3^2 & w_3^4 \end{pmatrix} \end{array}$$

$w_3^0 = 1$

$w_3^1 = e^{-j\frac{2\pi}{3}} = j$

$w_3^2 = e^{-j\frac{4\pi}{3}} = -1$

$w_3^4 = w_3^1 = j$

$$[W_3] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & -1 \\ 1 & -1 & j \end{pmatrix}$$

Twiddle Factor

Ex: $x(n) = \{0, 5, 1\}$, find the 3-point DFT

$\boxed{N = 3}$

DFT $X(K) \Longrightarrow X_3 = [W_3]_{3\times 3}\, x_3$

$$X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & -1 \\ 1 & -1 & j \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1.5 \\ 0.5+j \\ 0.5 \end{pmatrix}$$

$X(K) = \{1.5, 0.5+j, -0.5\}$

$|X(K)| = \{1.5, 1.11, 0.5\}$

$\angle X(K) = \{0, 63.4, 180^\circ\}$